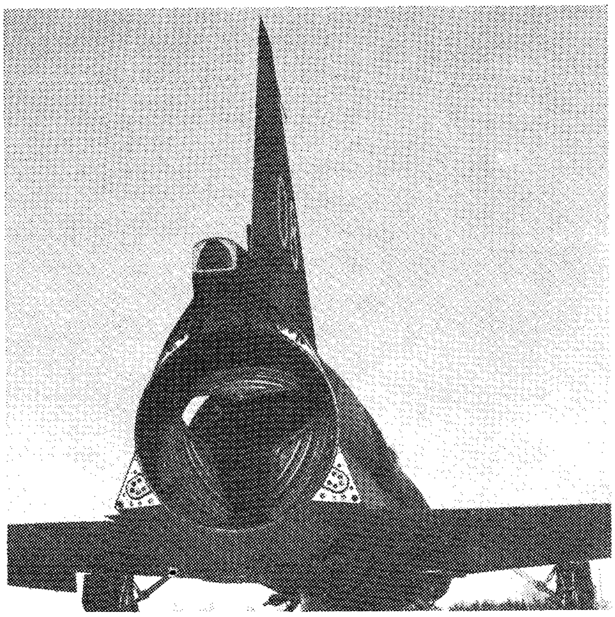

Fig. 1 Viggen thrust reverser in half-closed position.

to the afterbody by three supporting beams, one at each wing root and one below the fin. The slot between the afterbody and the leading edge of the ejector, divided into three parts by the beams, is the combined inlet for ambient air and nozzle for reverse exhaust gas. Instead of blow-in-doors, the slot has a translating sleeve which in the open position is retracted into the afterbody structure by hydraulic power. In the ejector shroud there are three blocker doors flush with the surface in the retracted position. When the reverser control is engaged, the doors are turned to their closed position by hydraulic actuators and the exhaust gas is deflected forward through the slot, forming one ground jet and two jets above the wing on either side of the fuselage. (See Fig. 1.) Thrust reversal can be preselected by the pilot in the air, and is then automatically initiated at touchdown of the main landing gear.

The development work on the reverser system included a number of activities. Besides model testing in wind tunnels and rigs and full-scale testing on the engine test bed, a large number of test runs (including landings) were carried out in prototype aircraft for evaluation of the complete reverser system and for demonstration of roll distance and aircraft stability at thrust reversal.

During thousands of reverser operations in the flight test program, invaluable but also hard-earned experience was gained. The problems encountered were mainly due to influences on aircraft pitch and yaw stability. Owing to the aerodynamic ground interference from the lower jet of the deflected flow, a strong variation in pitching moment with forward speed and degree of reverse thrust occurred. The yaw stability problem was also caused by aerodynamic interference. With the upper jets close to the fin, small asymmetric disturbances in the jet boundaries could result in side forces and yawing moments too great to be controlled by the pilot. After extensive analysis, testing, and simulation, various means to improve the stability were introduced and, finally, the aircraft behavior was quite satisfactory at all actual conditions.

The experience of the Viggen thrust reverser system from many years of service in the Swedish Air Force has been most favorable and the reverser has become a useful tool for the pilots, not only to shorten the roll distance but also in handling the aircraft during taxiing and parking. The mechanical functioning and durability of the reverser have been quite good and, in addition, a considerable savings of tires and brakes has been gained.

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Minimum Fuel Paths for a Subsonic Aircraft

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Nomenclature

A = aspect ratio
C_{D0} = zero incidence drag coefficient
C_L = lift coefficient
$C_{L\alpha}$ = slope of lift coefficient, per rad
C_1 = speed of sound, m/s
c = equivalent jet exhaust velocity, m/s = 3600 g/sfc
D = drag, N
$E(y)$ = ratio of density at altitude to sea level
F = function to be maximized
g = acceleration due to gravity, m/s^2
H = Hamiltonian
H_0 = reference altitude below tropopause ($T_0/l = 44.3$ km)
H_2 = scale height of isothermal atmosphere above tropopause (6343.2 m)
k = shape parameter for induced drag
K = $kC_{L\alpha}/\pi A$
l = lapse rate of temperature (6.5°C/km)
L = lift, N
M = mass, kg
M_f = final mass, kg
N = $\frac{1}{2}\rho_0 SC_{L\alpha}$, kg/m
D_0 = $\frac{1}{2}\rho_0 SC_{D0}$, kg/m
R = gas constant (287.423)
S = wing reference area, m^2
T = temperature, K
T_0 = sea level temperature (288 K)
T_2 = temperature above tropopause (216.5 K)
sfc = specific fuel consumption, kg/h/kg thrust
t = time, s
V = velocity, m/s
x = horizontal distance, m
y = altitude, m
α = incidence, rad
γ = climb angle, rad
Γ = ratio of specific heats of air ($C_p/C_v = 1.4$)
ρ = air density, kg/m^3
ρ_0 = sea-level air density, kg/m^3
μ = Mach number = V/C_1

Introduction

THE theory of optimal flight paths for winged or unwinged, supersonic rockets or aircraft has now reached a fairly complete stage. The history of optimal flight paths goes back to 1951, when Tsien and Evans[1] found the optimum thrust program for a vertically ascending sounding rocket. In 1952 Hibbs found the optimum burning program for horizontal flight of a winged supersonic rocket.[2]

Recently, the problem of the optimal climb path of a ballistic rocket away from the vertical has been shown in Ref. 4 to have a simple analytical solution. The problem of a winged, supersonic rocket, assuming the thrust acts along the flight path, has been solved in Ref. 5. The solution for a vectored, variable-thrust aircraft or missile has been given in Ref. 6. All of the solutions gave optimal flight paths which

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were supersonic from launch to thrust cutoff. Search of the literature reveals no equivalent success with the equivalent problem of subsonic flight.

This Note applies these modern methods of analysis to the subsonic case, and shows that a solution may be derived by a method which is a simple extension of that given in Ref. 5.

Equations of Motion

The equations for the aerodynamics of a subsonic aircraft are slightly more complicated than those of a supersonic aircraft. Supersonically, the resultant force due to lift acts normal to the body, so the drag and normal force are dependent on only two parameters, the zero-incidence drag coefficient and the normal force coefficient. Subsonically, the resultant force due to lift can act in a general direction, part way between the normal to the flight path (the lift direction) and the normal force direction (normal to the body axis). This means that, subsonically, the aerodynamics depends on a total of three parameters, the additional one being the coefficient of induced drag.

The coefficient of induced drag is given by

$$C_{Di}=k(C_L^2/\pi A) \tag{1}$$

where k is a shape parameter based on wing area S. The lift is given by

$$L=\tfrac{1}{2}\rho V^2 SC_L \tag{2}$$

If C_L is given by the usual linear relationship

$$C_L=C_{L\alpha}\alpha \tag{3}$$

then the drag D is given by

$$\begin{aligned} D&=\tfrac{1}{2}\rho V^2 S(C_{D0}+kC_{L\alpha}{}^2/\pi A\alpha^2) \\ &=\tfrac{1}{2}\rho V^2(SC_{D0}+KSC_{L\alpha}\alpha^2) \end{aligned} \tag{4}$$

where

$$K=(kC_{L\alpha})/\pi A \tag{5}$$

Let $\tfrac{1}{2}\rho_0 SC_{D0}=D_0$ and $\tfrac{1}{2}\rho_0 SC_{L\alpha}=N$. Then the total drag D is given by

$$D=(D_0+KN\alpha^2)V^2E(y) \tag{6}$$

Lift is given by

$$L=N\alpha V^2E(y) \tag{7}$$

$E(y)$ is the ratio of air density at altitude to that at sea level. For an isothermal atmosphere above the tropopause, $E(y)$ would be $\exp(-y/H_2)$, where H_2 is the scale height of the isothermal atmosphere.

The characteristics of the ICAO standard atmosphere are probably more familiar in Imperial units of lb·ft/s, but are in fact more simply given in metric units. Sea-level temperature of 288 K (15°C) decreasing linearly at 6.5°C per kilometer up to 11 km and thereafter remaining constant at all relevant altitudes for subsonic flight, at a temperature of 216.5 K (−56.5°C).

Temperature below 11 km altitude is given by

$$T=288\,(1-y/H_0) \tag{8}$$

where $H_0=(288/6.5)\times 1000=44.3$ km.

Density ratio is given by

$$\begin{aligned} E(y)&=\rho/\rho_0=(1-y/H_0)^{(gH_0/RT_0)-1} \\ &=(1-y/H_0)^{4.25} \end{aligned} \tag{9}$$

Above 11.0 km temperature is 216.5 K and the density ratio is given by

$$E(y)=\rho/\rho_0=0.29728\exp\left[-(y-11000)/H_2\right] \tag{10}$$

Density ratio at 11 km is 0.29728 of sea level and H_2, the scale height, is $RT_2/g=6343.2$ m.

The speed of sound, which is needed for evaluating Mach number, is given by

$$C_l=\sqrt{\Gamma RT}=\sqrt{\Gamma RT_0(1-y/H_0)} \tag{11}$$

where Γ is the ratio of specific heats of air, $C_p/C_v=1.40$, and R the gas constant = 287.423 (units).

Thrust of a jet engine is given by

$$T=-c(\mathrm{d}M/\mathrm{d}t) \tag{12}$$

where $-(\mathrm{d}M/\mathrm{d}t)$ is the rate of fuel burning.

c has the dimensions of velocity and is related to specific fuel consumption (sfc) by

$$c=3600\,g/\mathrm{sfc} \tag{13}$$

where c is the equivalent jet exhaust velocity and is dependent on velocity (or Mach number) and also on altitude through the thermodynamic efficiency of the engine. Initially, to find the form of the solution, it will be assumed that specific fuel consumption and the equivalent jet exhaust velocity c are constant. The modification to the solution of varying c will be given later.

The equations of motion for subsonic flight, neglecting Earth curvature, become

$$c\mathrm{d}M/\mathrm{d}t+M(\mathrm{d}V/\mathrm{d}t+g\sin\gamma)+(D_0(\mu)+KN\alpha^2)V^2E(y)=0 \tag{14}$$

$$M[V\mathrm{d}\gamma/\mathrm{d}t+g\cos\gamma]-N\alpha V^2E(y)=0 \tag{15}$$

$$\mathrm{d}y/\mathrm{d}t-V\sin\gamma=0 \qquad \mathrm{d}x/\mathrm{d}t-V\cos\gamma=0 \tag{16}$$

The drag coefficient is made a function of Mach number μ, while the effect of Mach number variation on lift is initially neglected. It will be shown that the transonic rise of drag "traps" the velocity to remain subsonic. We also need the relationship for Mach number given by

$$V-\mu C_l(y)=0 \tag{17}$$

Speed of sound C_l is a function of altitude y, through the temperature dependence with altitude [see Eq. (11)]. Above the tropopause, C_l is constant and $\partial C_l/\partial y=0$.

Originally it was thought that acceleration $\mathrm{d}V/\mathrm{d}t$ and path curvature $V\mathrm{d}\gamma/\mathrm{d}t$ were so small that the differential variables could be neglected, leading to simpler algebraic variables. However it is found that incorporating the complete equations leads to no essential complication, while neglect of the path curvature would lead to an actual singularity during cruise.

Equations (14-16) give only four differential equations between five differential variables namely M, V, γ, y, and x. One differential variable, V can be eliminated by a simple transformation of variables. Let

$$(1/M)\mathrm{d}M+(1/c)\,Vv=(1/\phi)\mathrm{d}\phi$$

then

$$\phi=M\exp(V/c) \tag{18}$$

ϕ is the characteristic mass, that is the mass required before an impulsive boost to give a final mass M and final velocity V. Characteristic mass ϕ is conserved through an impulsive boost and is continuous along a trajectory, although it is slowly dissipated by gravity and drag losses. The velocity V disappears as a differential variable and reappears as an algebraic (control) variable. This velocity can be freely chosen to satisfy the equation of optimal climb.

The equations become

$$\frac{1}{\phi}\frac{d\phi}{dt}+\frac{g}{c}\sin\gamma+(D_0(\mu)+KN\alpha^2)\frac{V^2}{\phi c}\exp\left(\frac{V}{c}\right)E(y)=0 \quad (19)$$

$$\frac{d\gamma}{dt}+\frac{g}{V}\cos\gamma-\frac{N\alpha V}{\phi}\exp\left(\frac{V}{c}\right)E(y)=0 \quad (20)$$

Using Hamiltonian or Pontryagin p_i undetermined multipliers, the equation for the minimal fuel, optimal trajectory must satisfy

$$\delta\int\left\{p_1\left[\frac{1}{\phi}\frac{d\phi}{dt}+\frac{g}{c}\sin\gamma+(D_0(\mu)+KN\alpha^2)\frac{V^2}{\phi c}\exp\left(\frac{V}{c}\right)E(y)\right]\right.$$
$$+p_2\left[\frac{d\gamma}{dt}+\frac{g}{V}\cos\gamma-\frac{N\alpha V}{\phi}\exp\left(\frac{V}{c}\right)E(y)\right]$$
$$+\frac{p_3}{H_2}\left[\frac{dy}{dt}-V\sin\gamma\right]+\frac{p_4}{H_2}\left[\frac{dx}{dt}-V\cos\gamma\right]$$
$$\left.+p_5\left[V-\mu C_1\right]\right\}dt=0 \quad (21)$$

Solution of the Optimal Equations

Equation (21) shows that there are four differential equations between four state variables ϕ, γ, y, and x, two control variables V and α, and one algebraic variable, Mach number μ which satisfies the equation of constraint. Thus velocity V rather than thrust controls the flight path longitudinally, while incidence controls the curvature of the flight path laterally.

For each differential (state) variable, there is an Euler-Lagrange equation

$$d/dt(\partial F/\partial\dot{x}_i)=\partial F/\partial x_i \qquad (x_i=\phi,\gamma,y,x) \quad (22)$$

For each control or constraint variable

$$\partial F/\partial u_i=0 \qquad (u_i=V,\alpha,\mu) \quad (23)$$

With respect to α,

$$p_1(2KN\alpha V^2)/\phi c-(p_2NV)/\phi=0$$

therefore

$$p_2=p_1(2K\alpha)(V/c) \quad (24)$$

Alternatively, and more important, this can be regarded as giving a solution for incidence α in terms of the other parameters

$$\alpha=p_2c/(2p_1KV) \quad (25)$$

With respect to ϕ

$$\frac{d}{dt}\left(\frac{p_1}{\phi}\right)=\frac{1}{\phi}\frac{dp_1}{dt}-\frac{p_1}{\phi^2}\frac{d\phi}{dt}=-\frac{p_1}{\phi^2}\frac{d\phi}{dt}-p_1(D_0(\mu)$$
$$+KN\alpha^2)\frac{V^2}{\phi^2c}\exp\left(\frac{V}{c}\right)E(y)+p_2\frac{N\alpha V}{\phi^2}\exp\left(\frac{V}{c}\right)E(y)$$

therefore

$$\frac{dp_1}{dt}=-p_1(D_0(\mu)-KN\alpha^2)\frac{V^2}{\phi c}\exp\left(\frac{V}{c}\right)E(y) \quad (26)$$

With respect to γ

$$\frac{dp_2}{dt}=p_1\frac{g}{c}\cos\gamma-p_2\frac{g}{V}\sin\gamma-p_3\frac{V}{H_2}\cos\gamma+\frac{p_4}{H_2}V\sin\gamma \quad (27)$$

With respect to y

$$\frac{1}{H_2}\frac{dp_3}{dt}=p_1(D_0(\mu)+KN\alpha^2)\frac{V^2}{\phi c}\exp\left(\frac{V}{c}\right)\frac{\partial E(y)}{\partial y}$$
$$-p_2\frac{N\alpha V}{\phi}\exp\left(\frac{V}{c}\right)\frac{\partial E(y)}{\partial y}-p_5\mu\frac{\partial C_1}{\partial y}$$

therefore

$$\frac{dp_3}{dt}=p_1(D_0(\mu)-KN\alpha^2)\frac{V^2}{\phi c}\exp\left(\frac{V}{c}\right)H_2\frac{\partial E(y)}{\partial y}$$
$$-p_5\mu H_2\frac{\partial C_1}{\partial y} \quad (28)$$

With respect to x

$$\frac{dp_4}{dt}=0, \quad \text{therefore} \quad p_4=C_4\ (\text{const}) \quad (29)$$

With respect to Mach number μ

$$p_1(\partial D_0/\partial\mu)(V^2/\phi c)\exp(V/c)E(y)-p_5C_1(y)=0$$

therefore

$$p_5=\frac{p_1}{c_1}\frac{\partial D_0}{\partial\mu}\frac{V^2}{\phi c}\exp\left(\frac{V}{c}\right)E(y) \quad (30)$$

With respect to the algebraic variable velocity V

$$p_1(D_0(\mu)+KN\alpha^2)\frac{V}{\phi c}(2+V/c)\exp\left(\frac{v}{c}\right)E(y)$$
$$-p_2\frac{N}{\phi}\alpha\left(1+\frac{V}{c}\right)\exp\left(\frac{V}{c}\right)E(y)-p_2\frac{g}{V^2}\cos\gamma$$
$$-\frac{p_3}{H_2}\sin\gamma-\frac{p_4}{H_2}\cos\gamma+p_5=0 \quad (31)$$

The Hamiltonian H is given by

$$H=\Sigma\dot{x}_i(\partial F/\partial\dot{x}_i)-F \quad (32)$$

and since H is explicitly independent of time t

$$H=\text{const}$$

Hence

$$H=-p_1(D_0(\mu)+KN\alpha^2)\frac{V^2}{\phi c}\exp\left(\frac{V}{c}\right)E(y)$$
$$+p_2\frac{N\alpha V}{\phi}\exp\left(\frac{V}{c}\right)E(y)-p_1\frac{g}{c}\sin\gamma-p_2\frac{g}{V}\cos\gamma$$
$$+\frac{p_3}{H_2}V\sin\gamma+p_4\frac{V}{H_2}\cos\gamma \quad (33)$$

Multiplying Eq. (31) by V and adding H, Eq. (33) gives

$$p_1\left[D_0(\mu)\left(1+\frac{V}{c}\right)+KN\alpha^2\left(1-\frac{V}{c}\right)\right]\frac{V^2}{\phi c}\exp\left(\frac{V}{c}\right)E(y)$$

$$+p_1\mu\frac{\partial D_0}{\partial\mu}\frac{V^2}{\phi c}\exp\left(\frac{V}{c}\right)E(y)-p_1\frac{g}{c}\sin\gamma-2p_2\frac{g}{V}\cos\gamma=H$$

Simplifying this gives

$$\left[D_0(\mu)\left(1+\frac{V}{c}\right)+\mu\frac{\partial D_0}{\partial\mu}+KN\alpha^2\left(1-\frac{V}{c}\right)\right]V^2\exp\left(\frac{V}{c}\right)E(y)$$

$$=\phi g\left(\sin\gamma+4K\alpha\cos\gamma+\frac{cH}{p_1 g}\right) \tag{34}$$

Equation (34) for velocity and Eq. (25) for incidence are the two basic algebraic equations that must be satisfied simultaneously all along the flight path. These equations are nonlinear and must be solved by iteration. The other equations, four for the equations of motion and four for the adjoint Hamiltonian variables, p_i, must be integrated along the trajectory defined by the solution of Eqs. (25) and (34).

The variation terms outside the integral become

$$\left[p_1\frac{\Delta\phi}{\phi}+p_2\Delta\gamma+\frac{p_3}{H_2}\Delta y+\frac{p_4}{H_2}\Delta x-H\Delta t\right]_1^2=0 \tag{35}$$

Equation (35), if the adjoint variables p_i are known, gives the tradeoff between fuel consumption, velocity, and trajectory variables. For the fuel consumption to be a true minimum, independent of time, the Hamiltonian should be identically zero.

Analysis of the Cruise

Cruise is usually stated to occur at optimum lift/drag ratio. This begs the question that lift/drag ratio is not unique, being a continuous function of Mach number. Lift/drag ratio is given by

$$L/D=N\alpha/(D_0(\mu)+KN\alpha^2) \tag{36}$$

This has a maximum where

$$KN\alpha^2=D_0(\mu) \qquad \alpha(\mu)=\sqrt{D_0(\mu)/KN} \tag{37}$$

$$L/D=N/[2D_0(\mu)]\sqrt{D_0(\mu)/KN}=\tfrac{1}{2}\sqrt{N/[KD_0(\mu)]} \tag{38}$$

Inspection of the adjoint Eqs. (26-31) shows that p_1 is constant and, above the tropopause, p_3 is constant. If Mach number is constant, then incidence and velocity are also constant, hence p_2 is constant. Thus all the adjoint parameters are constant.

Multiply the fuel consumption equation by V and use

$$N\alpha V^2E(y)=Mg\cos\gamma=Mg\dot{x}/V \tag{39}$$

Then

$$cV/M\,\mathrm{d}M/\mathrm{d}t+V\dot{V}+g\dot{y}+(D/L)g\dot{x}=0$$

Integrating at constant L/D gives

$$RA=x+\left(\frac{L}{D}\right)\left[y+\frac{(V^2-VL^2)}{2g}+\frac{Vc}{g}\log\left(\frac{M}{M_f}\right)\right] \tag{40}$$

where the individual terms can be recognized as range gone, range due to glide from altitude, range due to kinetic energy $V^2/2$, and range to go during cruise. Since (L/D) decreases with Mach number, then Eq. (40) for range has a definite maximum, provided (L/D) decreases faster than $V^2/2g$ increases. The velocity is then trapped subsonically.

Table 1 Input parameters

M_0	$=1.5\times10^5$ kg
sfc	$=0.6$ kg/h/kg thrust
$\frac{1}{2}\rho_0 SC_{D0}$	$=3.0$
$\frac{1}{2}\rho_0 SC_{L\alpha}$	$=800$
K	$=0.25\ (k\simeq1)$
A	$=8$
$D_0(\mu)$	$=D_0\ (1+\mu^6)$

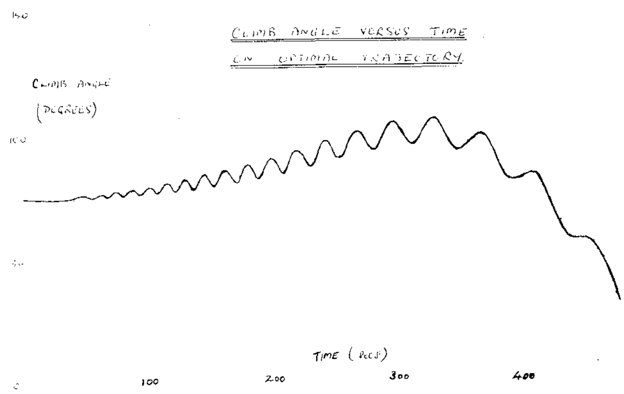

Fig. 1 Climb angle vs time on optimal trajectory.

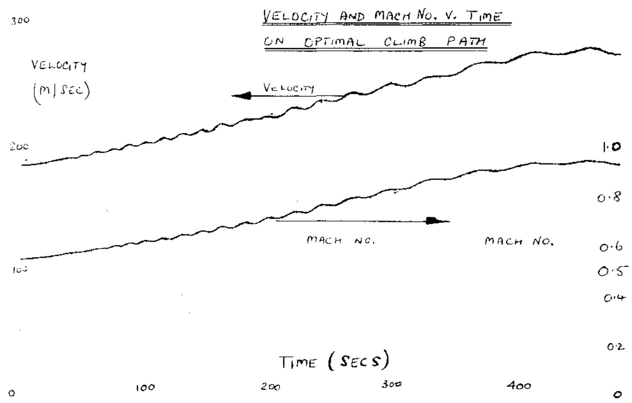

Fig. 2 Velocity and Mach number vs time on optimal climb path.

Using Eqs. (40) and (38), it is simple enough to evaluate the Mach number and hence velocity at which the maximum occurs. For practical drag rises with Mach number, this usually occurs at about Mach 0.9.

The Hamiltonian is identically zero, and we have

$$p_2=p_1 2K(\alpha V)/c=\text{const}$$

$$p_3=p_1(gH_2/Vc)=\text{const}$$

$$p_4 \text{ and } p_1=\text{const}$$

$$p_4=p_2(gH_2/V^2)=\text{const}$$

Since range to go during cruise and range to go due to kinetic energy have different variation with velocity, then Eq. (40) indicates that the maximum will vary along the flight path according to the range to go. At the end when $(Vc/g)\log(M/M_f)$ is small, the velocity is determined by the kinetic energy and will be greater than when the range to go is large.

Computation of the Climb to Altitude

Since cruise at altitude and descent are so easily dealt with by operating at the optimum lift/drag ratio appropriate to the Mach number of cruise or descent, the only problem

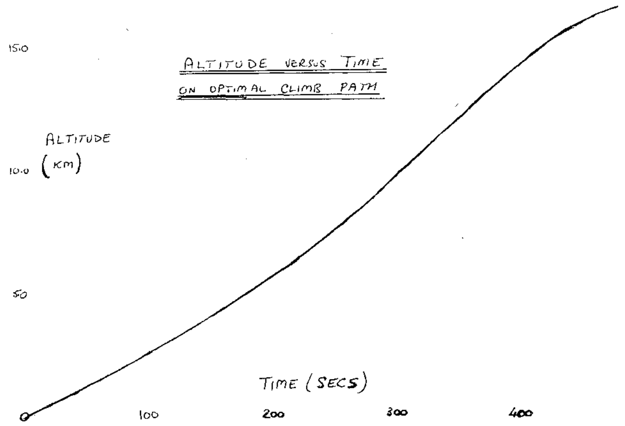

Fig. 3 Altitude vs time on optimal climb path.

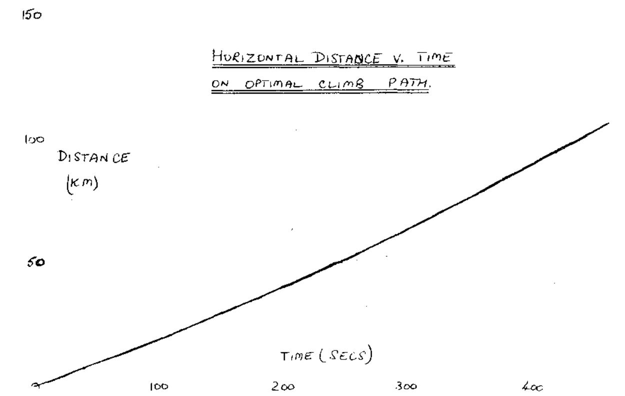

Fig. 4 Horizontal distance vs time on optimal climb path.

remaining is that of the climb to altitude. This involves the simultaneous solution of the equations for velocity and incidence [Eqs. (25) and (34)], together with integration of the equations of motion and constraint [Eqs. (16-20)] and the equations for the Hamiltonian adjoint variables [Eqs. (26-30)].

The equations have been programmed onto an interactive digital computer. Visual display of results on a VDU is desirable for quick and easy assessment of the climb path. The program is quite short in terms of program length and computer storage. The iteration for velocity and incidence uses a simple Newton-Raphson, first-order approximation, and the integration routine was a simple trapezoidal, although fourth-order Runge-Kutta is to be recommended.

The aircraft parameters incorporated were those appropriate to a wide-bodied jumbo jet, of diameter 5 m, length about 75 m, and wing area 200 m^2. Specific fuel consumption was taken as 0.6 kg/h/kg thrust. Drag rise with Mach number was taken as a simple power law with an exponent of about 6 to give a rapid rise of drag with Mach number at around Mach 0.9. Details of the inputs are listed in Table 1, giving a doubling of profile drag at Mach 1.0.

The computation was comparatively well behaved, although not as well behaved as the supersonic case. Oscillations of the flight path about the mean tended to occur if the initial conditions were not suitably chosen. Time period of the oscillations was 20-25 s, giving about 15 oscillations over the whole ascent phase which lasted about 400 s over a horizontal distance of 100 km.

Initial climb angle was 7.5 deg and initial velocity 200 m/s. Initial incidence was determined from the normal force required to give zero path curvature, $\dot{\gamma}_i = 0$. Figures 1-4 show the characteristics of the climb path. Figure 1 shows the flight path angle and Fig. 2 gives the velocity and Mach number. Unlike a normal calculation of trajectory, where the velocity falls off as the path angle increases, for the optimal path the velocity should increase as path angle increases to minimize the gravity loss. Figure 3 shows the altitude against time and Fig. 4 horizontal distance against time.

Other vehicles have been evaluated, and are also comparatively well behaved, but the details are so similar to those above that they add little to the visualization of the problem. The method works well in practice, and little remains except to calculate climb paths with greater and greater accuracy. The aircraft should accelerate as rapidly as possible from takeoff to 200 m/s, and be held down until rotation up to 7.5 deg is achieved at 200 m/s.

Allowance for Variation of sfc with Velocity

Due to the increased thermodynamic efficiency of ram compression at high velocity, it is likely that specific fuel consumption will decrease and equivalent jet exhaust velocity c will increase with velocity. This can quite easily be incorporated into the present theory.

Let $c = c(V)$. Define a new characteristic mass ϕ by

$$\mathrm{d}\phi/\phi = (1/M)\,\mathrm{d}M + \mathrm{d}v/c(v)$$

$$\phi = M\exp\int_0^v \mathrm{d}v/c(v) \qquad (41)$$

Then all the equations follow as before, except that the longitudinal equation includes two more terms containing $(V/c)(\partial c/\partial V)$. The longitudinal equation is now not so simple, since it does not now consist only of drag terms containing velocity and other terms which do not contain velocity. However the terms containing $(V/c)\partial c/\partial V$ are small and do not perturb the result significantly.

If specific fuel consumption depends on altitude as well as velocity, the foregoing theory can not be applied. Above the tropopause, there is no thermodynamic reason why fuel consumption should be inherently dependent on altitude, so the above theory should be immediately applicable. Where sfc does depend on altitude, it is impossible to define a single-valued characteristic mass; rate of fuel consumption $\dot{m}$ and acceleration $\dot{V}$ must be kept as separate variables. There is, however, an additional adjoint equation which could, in principle, be used to eliminate the differential variables, leaving two algebraic equations for velocity and incidence as before. Extension of the theory to this case has not yet been developed.

Conclusion

A method has been derived for finding the optimal flight path which minimizes fuel consumption for a subsonic aircraft. The method is basically a simple extension of that used in Ref. 5 for supersonic aircraft or missiles.

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